FREE CONVECTION IN AN IMMERSED PLANE JET

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We investigate the free convection in a plane immersed jet by the method of self-similar and nonself-similar solutions. The derived solutions are valid both for a heated and a cooled jet.

Let us examine the flow of a plane laminar jet directed vertically upward from a long narrow slit into a space containing the same nonmoving gas. We will assume that the jet impinges with some momentum J_0 [1]. The free convection produced by the temperature difference in the jet leads to additional motion. Given a small temperature difference between the jet and the ambient medium, for this problem we can use the boundary-layer equations, introducing the additional force associated with the existing temperature difference into the equation of motion.

According to [2], the basic equations will have the following form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta T_\infty\Theta, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

$$u\frac{\partial \Theta}{\partial x}+v\frac{\partial \Theta}{\partial y}=a\frac{\partial^2 \Theta}{\partial y^2}. \tag{3}$$

The boundary conditions are the usual ones [1]

$$\frac{\partial u}{\partial y}=0,\ \frac{\partial \Theta}{\partial y}=0 \quad \text{for } y=0, \qquad u=0,\quad \Theta=0 \quad \text{as } y\to\infty. \tag{4}$$

Using the boundary equations, we introduce the stream function, so that (1) and (3) are written in the following form:

$$\frac{\partial\Psi}{\partial y}\frac{\partial^2\Psi}{\partial x\partial y}-\frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial y^2}=\nu\frac{\partial^3\Psi}{\partial y^3}+g\beta T_\infty\Theta, \tag{5}$$

$$\frac{\partial\Psi}{\partial y}\frac{\partial\Theta}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\Theta}{\partial y}=a\frac{\partial^2\Theta}{\partial y^2}. \tag{6}$$

1. In seeking the self-similar solution, we will present the stream function and the dimensionless excess temperature in the form

$$\Psi=\chi(x)f(\eta),\quad \Theta=p(x)\varphi(\eta),\quad \eta=\omega(x)y. \tag{7}$$

Having substituted (7) into system (5) and (6), we obtain

$$\omega\chi(\omega\chi)'f'^2-\omega^2ff''\chi\chi'=\nu\chi\omega^3f'''+g\beta T_\infty p\varphi, \tag{8}$$
$$\chi p'f'\varphi-\chi'pf\varphi'=ap\omega\varphi'', \tag{9}$$

where the primes denote differentiation with respect to the argument of the given function.

Let us first attempt to determine the generalized self-similar solution [3]. It can be demonstrated that such a solution exists only for Pr = 2. Here we should bear in mind that the resulting solution for a

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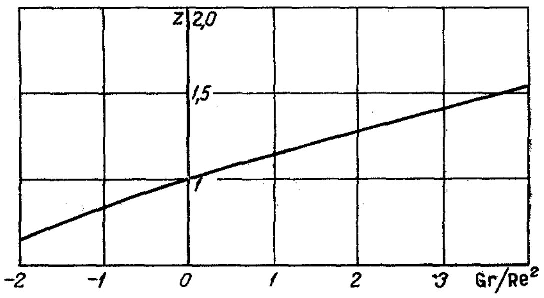

Fig. 1. Graph showing z as a function of Gr/Re^2.

vanishingly small temperature difference must become the solution for a jet without free convection [1]. It is therefore natural to assume that

$$f = \operatorname{th}\eta, \quad \varphi = \operatorname{sech}^4\eta.$$

Then, to determine ω, χ, and p, we obtain the following system of ordinary differential equations:

$$\begin{gathered}\chi p' = -4ap\omega, \quad \chi p' + 4\chi' p = 20\, ap\omega, \\ \omega\chi(\omega\chi)' - g\beta T_\infty p = -2\nu\chi\omega^3.\end{gathered} \tag{10}$$

After simple transformation, we obtain for the solution of system (10)

$$\begin{gathered}\frac{\chi}{2a\sqrt{\nu}} x^{-1/3} = z^{1/4}, \quad \frac{3}{a}\frac{\overline{\nu}}{}\omega x^{2/3} = \frac{\left(1 + \frac{3}{4}\frac{\mathrm{Gr}}{\mathrm{Re}^2} z\right)^{1/3}}{z^{1/2}}, \\ \frac{\omega\chi}{\frac{2}{3}a^2} x^{1/3} = \frac{\left(1 + \frac{3}{4}\frac{\mathrm{Gr}}{\mathrm{Re}^2} z\right)^{1/3}}{z^{1/4}}, \quad \frac{p}{\frac{15}{32}\frac{H_t}{T_\infty a\sqrt{\nu}}} x^{1/3} = z^{-1/4},\end{gathered} \tag{11}$$

where

$$\int_0^{z_1} z_1^{-1/4}(1 + z_1)^{1/3} dz_1 = \left(\frac{4}{3}\right)^{1/4}\left(\frac{\mathrm{Gr}}{\mathrm{Re}^2}\right)^{3/4},$$

$$z = z_1\frac{4}{3}\left(\frac{\mathrm{Gr}}{\mathrm{Re}^2}\right)^{-1}, \quad a = \left(\frac{9}{16}\frac{J_0}{\rho\sqrt{\nu}}\right)^{1/2}.$$

The numerical results are shown in Figs. 1 and 2. In analyzing (11) we must consider two cases. In the first, the temperature of the medium is higher than the temperature of the jet ($Gr/Re^2 < 0$). The jets are decelerated under the action of free convection, and there exists a section in which the jet is completely stagnated ($Gr/Re^2 = -2.09$). In the second case, the jet temperature is higher than that of the ambient medium. As a consequence of convection, the penetrating power of the jet is increased.

With a large value for Gr/Re^2, in the integral following (11) we can neglect unity as small in comparison with z_1, and we can represent the expression for z_1 in the explicit form

$$z_1 \cong \left(\frac{5}{12}\right)^{\frac{12}{5}}\left(\frac{4}{3}\right)^{\frac{3}{5}}\left(\frac{\mathrm{Gr}}{\mathrm{Re}^2}\right)^{\frac{9}{5}}. \tag{12}$$

An exponential relationship is found between the basic quantities and x in (11) in this case. Thus we see from this special example that the solutions derived in [2] are applicable only at a great distance from the source. However, reference [2] contains an inaccuracy in the determination of the constants. It can be demonstrated (in the notation of [2]) that in this problem there is only one constant a which has to be determined from the constancy of H_t in the jet, i.e.,

$$a^5 = \frac{2H_t g\beta}{\nu^5\int_0^\infty f' p\, d\eta}. \tag{13}$$

The resulting basic relationships (Pr = 2) derived from [2] in the determination of a from (13) coincide with (11) in the determination of z_1 from (12) and with the use of the condition $z_1 \gg 1$.

2. There is considerable interest in an examination of free convection for other values of the Pr number; this is true particularly for air (Pr = 0.72). Here it is natural to turn to the nonself-similar methods of solution.

Since there is generally some interest in the limited effect exerted by free convection on the discharging jet, we can propose a method of solution that is close to the self-similar [4]. Let us present the stream function $\Psi(x, y)$ and the excess temperature difference $\Theta(x, y)$ in the form of the series

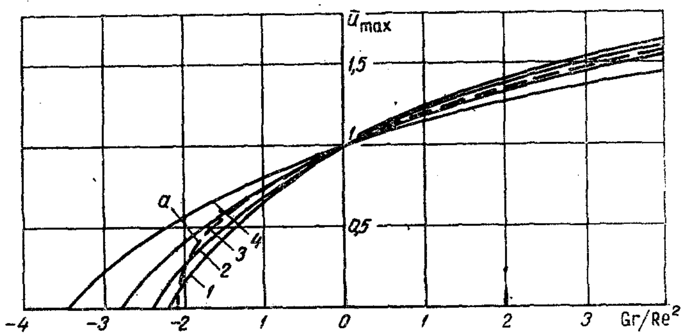

Fig. 2. Graph showing $\bar{u}_{max}$ as a function of Gr/Re^2: 1) Pr = 0.72; 2) 1.0; 3) 2; 4) 5; a) exact solution for Pr = 2.

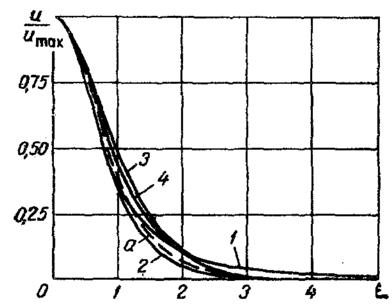

Fig. 3

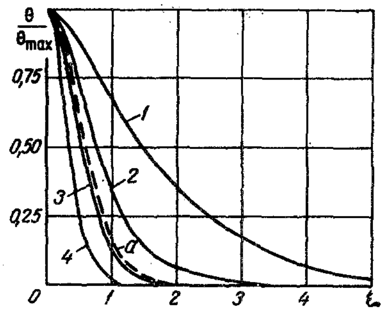

Fig. 4

Fig. 3. Graph showing u/u_{max} as a function of ξ: 1) Pr = 0.72; 2) 1.0; 3) 2; 4) 5; a) exact solution for Pr = 2.

Fig. 4. Graph showing Θ/Θ_{max} as a function of ξ: 1) Pr = 0.72; 2) 1.0; 3) 2; 4) 5; a) exact solution for Pr = 2.

TABLE 1. Values of the Basic Jet Characteristics

Pr	0,72			1,0		
n	0	1	2	0	1	2
$F_n(\infty)$	1,0000	0,2522	0,4312	1,0000	0,1250	—0,0768
$F'_n(0)$	1,0000	0,2678	—0,0506	1,0000	0,2500	—0,0417
$\tau_n(0)$	1,0000	—0,0758	0,0263	1,0000	—0,0538	0,0105
Pr	**2,0**			**5,0**		
n	0	1	2	0	1	2
$F_n(\infty)$	1,0000	0,0357	0,0006	1,0000	0,1000	—0,0001
$F'_n(0)$	1,0000	0,2143	—0,0307	1,0000	0,1687	—0,0203
$\tau_n(0)$	1,0000	—0,0357	0,0498	1,0000	—0,0259	0,0029

$$\Psi(x,\ y) = 2\alpha\sqrt{\nu}\,x^{1/3}\sum_{n=0}^{\infty}\left(\frac{\mathrm{Gr}}{\mathrm{Re}^2}\right)^n F_n(\xi),$$
$$\Theta(x,\ y) = \frac{H_t}{T_\infty \alpha\sqrt{\nu}}\ \frac{1}{2}\,\mathrm{B}^{-1}\left(\frac{1}{2},\ 1+\mathrm{Pr}\right)\sum_{n=0}^{\infty}\left(\frac{\mathrm{Gr}}{\mathrm{Re}^2}\right)^n \tau_n(\xi). \qquad (14)$$

Substituting these series into (5) and (6), and equating the coefficients for identical powers of Gr/Re^2, we find a system of ordinary differential equations for the determination of $F_n(\xi)$ and $\tau_n(\xi)$

$$\sum_{k=0}^{n} [(-1+4k)F'_{n-k}F'_k - (1+4k)F_kF''_{n-k}] = \frac{1}{2}F'''_n - \tau_{n-1},$$

$$\sum_{k=0}^{n} [(-1+4k)F'_{n-k}\tau_k - (1+4k)F_k\tau'_{n-k}] = \frac{1}{2\mathrm{Pr}}\tau''_n. \qquad (15)$$

For $F_0(\xi)$ and $\tau_0(\xi)$ we find solutions from the corresponding self-similar problem [1] without free convection:

$$F_0(\xi) = \mathrm{th}\,\xi, \quad \tau_0(\xi) = \mathrm{sech}^{2\mathrm{Pr}}\,\xi.$$

According to the general method outlined in [5] for problems of free jet discharge, the system of equations (15) can be reduced to a system of successively solved Legendre equations. The solution is then found in quadratures as the particular integral of the corresponding nonuniform equation, in particular for Pr = 1:

$$F_1 = \frac{1}{8}(F_0 + \xi F'_0),$$

$$\tau_1 = \frac{1}{8}\left(2F'_0 + \xi F''_0\right) - \frac{1}{3}\left[1 - \frac{\pi}{2}\sqrt{1-F_0^2}\,\frac{P'_\mu(F_0) - P'_\mu(-F_0)}{\mathrm{ch}\,\pi\tau}\right], \qquad (16)$$

where

$$\mu = -\frac{1}{2} + i\,\frac{\sqrt{23}}{2}, \quad \tau = \frac{\sqrt{23}}{2}.$$

The numerical solution of Eqs. (15) was undertaken for various values of Pr, and we obtained the first two approximations. Figures 3 and 4 show the graphs for u/u_{max} and Θ/Θ_{max}. In Fig. 2 we compare the derived exact solution with the approximate solutions. Table 1 gives the numerical values of the basic characteristics of the jet.

Because of the impossibility of calculating a large number of terms in series (14), there is some interest in applying the nonlinear Shanks [6] transform to the first three terms of the exponential series (14). The calculations for $\bar{u}_{max}$ were accomplished with such a transformation, and it yielded better agreement for Pr = 2 with the exact solution than arithmetic summation (Fig. 2).

NOTATION

x and y	are Cartesian coordinates;
$\xi = \alpha y x^{-2/3}/3\sqrt{\nu}$	is a dimensionless self-similar coordinate;
u and v	are velocity components in the directions of the x- and y-axes;
ν	is the coefficient of kinematic viscosity;
g	is the acceleration of the force of gravity;
β	is the coefficient of thermal expansion;
a	is the coefficient of thermal diffusivity;
T_∞	is the temperature of the external medium;
$T - T_\infty/T_\infty = \Theta$	is the dimensionless excess temperature;
$H_T = 2\int_0^\infty u(T - T_\infty)dy$	is the excess heat content;
$\mathrm{Pr} = \nu/a$	is the Prandtl number;
$\mathrm{Gr} = (g\beta\delta^3/\nu^2)\Delta T$	is the Grashof number;
$\mathrm{Re} = u_1\delta/\nu$	is the Reynolds number;
$u_1 = 2\alpha^2 x^{-1/3}/3$	is the characteristic velocity in similarity criteria;
$\delta = 3\sqrt{\nu}x^{2/3}/\alpha$,	is the characteristic dimension m similarity criteria;
$\Delta T = (H_t/2\nu)B^{-1}$	
(1/2, 1 + Pr)	is the characteristic temperature difference in similarity criteria;
$B(a, b)$	is the beta function.

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